

REGULARITY CRITERION FOR 3D NAVIER-STOKES EQUATIONS IN BESOV SPACES

DAOYUAN FANG CHENYIN QIAN*
DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY,
HANGZHOU, 310027, CHINA

ABSTRACT. Several regularity criterions of Leray-Hopf weak solutions u to the 3D Navier-Stokes equations are obtained. The results show that a weak solution u becomes regular if the gradient of velocity component $\nabla_h u$ (or ∇u_3) satisfies the additional conditions in the class of $L^q(0, T; \dot{B}_{p,r}^s(\mathbb{R}^3))$, where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient operator. Besides, we also consider the anisotropic regularity criterion for the weak solution of Navier-Stokes equations in \mathbb{R}^3 . Finally, we also get a further regularity criterion, when give the sufficient condition on $\partial_3 u_3$.

1. INTRODUCTION

In the present paper, we address sufficient conditions for the regularity of weak solutions of the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ u(x, 0) = u_0, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $u = (u_1, u_2, u_3) : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}$ is a scalar pressure, and u_0 is the initial velocity field, $\nu > 0$ is the viscosity. We set $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ as the horizontal gradient operator and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ as the horizontal Laplacian, and Δ and ∇ are the usual Laplacian and the gradient operators, respectively. Here we use the classical notations

$$(u \cdot \nabla)v = \sum_{i=1}^3 u_i \partial_{x_i} v_k, \quad (k = 1, 2, 3), \quad \nabla \cdot u = \sum_{i=1}^3 \partial_{x_i} u_i,$$

and for sake of simplicity, we denote ∂_{x_i} by ∂_i .

It is well known that the weak solution of the Navier-Stokes equations (1.1) is unique and regular in two dimensions. However, in three dimensions, the regularity problem of weak solutions of Navier-Stokes equations is an outstanding open problem in mathematical fluid mechanics. Strong solutions are known to exist for a short interval of time whose length depends on the initial data. Moreover, this strong solution is known to be unique and to depend continuously on the initial data (see, for example, [22], [24]). Let us recall the definition of Leray-Hopf weak solution. We set

$$\mathcal{V} = \{\phi : \text{ the 3D vector valued } C_0^\infty \text{ functions and } \nabla \cdot \phi = 0\},$$

which will form the space of test functions. Let H and V be the closure spaces of \mathcal{V} under L^2 -topology, and under H^1 -topology, respectively.

For $u_0 \in H$, the existence of weak solutions of (1.1) was established by Leray [15] and Hopf in [9], that is, u satisfies the following properties:

- (i) $u \in C_w([0, T]; H) \cap L^2(0, T; V)$, and $\partial_t u \in L^1(0, T; V')$, where V' is the dual space of V ;

2000 Mathematics Subject Classification. 35Q30; 76D05.

Key words. 3D Navier-Stokes equations; Leray-Hopf weak solution; Regularity criterion.

*E-mail addresses: dyf@zju.edu.cn (D. Fang), qcyjcsx@163.com (C. Qian).

(ii) u verifies (1.1) in the sense of distribution, i.e., for every test function $\phi \in C^\infty([0, T]; \mathcal{V})$, and for almost every $t, t_0 \in (0, T)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} u(x, t) \cdot \phi(x, t) dx - \int_{\mathbb{R}^3} u(x, t_0) \cdot \phi(x, t_0) dx \\ &= \int_{t_0}^t \int_{\mathbb{R}^3} [u(x, s) \cdot (\phi_t(x, s) + \nu \Delta \phi(x, s))] dx ds \\ &+ \int_{t_0}^t \int_{\mathbb{R}^3} [(u(x, s) \cdot \nabla) \phi(x, s)] \cdot u(x, s) dx ds \end{aligned}$$

(iii) The energy inequality, i.e.,

$$\|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_{t_0}^t \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2,$$

for every t and almost every t_0 .

It is well known, if $u_0 \in V$, a weak solution becomes strong solution of (1.1) on $(0, T)$ if, in addition, it satisfies

$$u \in C([0, T]; V) \cap L^2(0, T; H^2) \text{ and } \partial_t u \in L^2(0, T; H).$$

Researchers are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that the weak solutions become regular, and the first result is usually referred as Prodi-Serrin conditions (see [20] and [21]), which states that if a weak solution u is in the class of

$$u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 1, \quad s \in [3, \infty],$$

then the weak solution becomes regular.

The full regularity of weak solutions can also be proved under alternative assumptions on the gradient of the velocity ∇u . Specifically (see [3]), if

$$\nabla u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 2, \quad s \in [\frac{3}{2}, \infty]. \quad (1.2)$$

H. Beirão da Veiga [2] extended Serrin's regularity criterion to the vorticity $w = \text{curl} u$ showing that if

$$\omega \in L^q(0, T; L^r) \text{ with } \frac{2}{q} + \frac{3}{r} = 2, \quad \frac{3}{2} < r < \infty, \quad (1.3)$$

then u is regular. In the marginal case $r = \infty$, H. Kozono and Y. Taniuchi [10] proved the regularity of weak solutions under the condition

$$\omega \in L^1(0, T; BMO), \quad (1.4)$$

where BMO is the space of bounded mean oscillation defined by

$$f \in L_{loc}^1(\mathbb{R}^3), \quad \sup_{x, R} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \bar{f}_{B_R(x)}| dy < \infty,$$

where $\bar{f}_{B_R(x)}$ is the average of f over $B_R(x) = \{y \in \mathbb{R}^3; |x - y| < R\}$.

Recently, the study of the regularity of weak solution involving Besov space becomes popular. For example, by establishing the logarithmic Sobolev inequality in Besov spaces, H. Kozono, T. Ogawa and Y. Taniuchi [11] refined the above two conditions to

$$\omega \in L^q(0, T; \dot{B}_{r, \infty}^0) \text{ with } \frac{2}{q} + \frac{3}{r} = 2, \quad 3 \leq r \leq \infty. \quad (1.5)$$

Here and thereafter, $\dot{B}_{p, q}^s$ stands for the homogeneous Besov space, see Section 2 for the definition. On the other hand, Chen and Zhang in [5] proved regularity criterion by imposing only

the two-component vorticity field. More precisely, they proved the regularity of weak solutions in the class of

$$\tilde{\omega} \in L^q(0, T; \dot{B}_{r, \sigma}^0) \text{ with } \tilde{\omega} = (\omega_1, \omega_2, 0), \frac{2}{q} + \frac{3}{r} = 2, \frac{3}{2} < r \leq \infty, \sigma \leq \frac{2}{3}. \quad (1.6)$$

In [12], H. Kozono and Y. Yatsu showed that if the Leray-Hopf weak solution u of (1.1) satisfies

$$\tilde{\omega} \in L^1(0, T; BMO), \quad (1.7)$$

then u is regular. More generally, B. Yuan and B. Zhang in [25] prove the weak solution u became regular if ω satisfies

$$\omega \in L^{\frac{2}{2-\alpha}}(0, T; \dot{B}_{\infty, \infty}^{-\alpha}(\mathbb{R}^3)) \cap L^{\frac{2}{1-\alpha}}(0, T; \dot{B}_{\infty, \infty}^{-1-\alpha}(\mathbb{R}^3)) \text{ for } 0 < \alpha < 1.$$

As to the endpoint case, S. Gala in [8] showed that if

$$\omega \in L^2(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)),$$

then the solution u was regular.

We point out that H. Kozono, T. Ogawa and Y. Taniuchi in [11] (Theorem 3.5) also got the full regularity of weak solutions under alternative assumptions on the velocity u . More precisely, if u satisfies

$$u \in L^2(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^3)), \quad (1.8)$$

then the solution u is regular. More generally, A. Cheskidov and R. Shvydkoy in [6] proved that the solution becomes smooth if

$$u \in L^r((0, T); B_{\infty, \infty}^{\frac{2}{r}-1}), \quad (1.9)$$

for some $r \in (2, \infty)$, where $B_{p, q}^s$ stands for the nonhomogeneous Besov space (for detail see [1]).

Motivated by the mentioned above, in this article, we consider assumptions on the gradient of velocity ∇u or the gradient of velocity component $\nabla_h u$ (or ∇u_3).

Our main results can be stated in the following:

Theorem 1.1. *Let u be a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1) with the initial value $u_0 \in V$. Suppose the vorticity $\omega = \text{curl} u$ satisfies the condition*

$$\nabla u \in L^2(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)). \quad (1.10)$$

Then u is regular.

Remark 1.2. By the definition of Besov space and Bernstein inequality (see (2.2) in section 2), we have

$$C^{-1} \|u\|_{\dot{B}_{\infty, \infty}^0} \leq \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} \leq C \|u\|_{\dot{B}_{\infty, \infty}^0}. \quad (1.11)$$

From (1.11), it is obvious that condition the (1.10) is equivalent to (1.8). Therefore, Theorem 1.1 is easy to get from the result of H. Kozono, T. Ogawa and Y. Taniuchi (Theorem 3.5) in [11].

Theorem 1.3. *Let u_0 and u be as in Theorem 1.1. Suppose that one of the following conditions is true:*

(i) the Leray-Hopf weak solution u satisfies

$$\nabla_h u \in L^{\frac{8}{3}}(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)). \quad (1.12)$$

(ii) u satisfies the following condition

$$\nabla u_3 \in L^{\frac{8}{5-2s}}(0, T; \dot{B}_{\infty, \infty}^{-s}(\mathbb{R}^3)) \text{ with } 0 < s < 1. \quad (1.13)$$

Then u is regular.

Corollary 1.4. *Suppose that $u_0 \in V$, and u is a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1). If u satisfies the condition*

$$u \in L^{\frac{8}{3}}(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^3)), \quad (1.14)$$

or, for any (small) positive real number ε , satisfies

$$u_3 \in L^{\frac{8}{5-2s}}(0, T; \dot{B}_{\infty, \infty}^{-s+1}(\mathbb{R}^3)) \text{ with } 0 < s < 1. \quad (1.15)$$

Then u is regular.

Remark 1.5. Theorem 1.3 pays attention to the case of the gradient of velocity component, the first result of it proves the regularity criterion by imposing only the two-component of the gradient of velocity field, namely the horizontal gradient components. By (1.11), the Corollary 1.4 is easy to get from Theorem 1.3, and we see that the first result of Corollary 1.4 is also a consequence of (1.8) or Theorem 1.1. However, the regularity result in case of $\nabla_h u$ satisfies (1.12) is not a trivial corollary of Theorem 1.1.

In framework of the Lebesgue spaces, the regularity criterion problem has been in-deep study with the conditions in terms of one component ∇u_3 (for example, see [26], [19]) or one directional derivative $\partial_3 u$ (for example, see [18], [13]). Because of the embedding

$$L^p(\mathbb{R}^3) \hookrightarrow \dot{B}_{r, q}^{\frac{3}{r} - \frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty, \infty}^{-\frac{3}{p}}(\mathbb{R}^3), \quad (1.16)$$

with $2 \leq p < r, q \leq \infty$, a natural idea is to extend the regularity criterion results to the framework of Besov spaces. To our knowledge, there are some results in term of the whole velocity or the vorticity. Motivated by the literature [18], [13], we want to consider additional condition only on $\partial_1 u_3, \partial_2 u_3, \partial_3 u_3$ instead of ∇u in the more general spaces. Our result reads as:

Theorem 1.6. *Let u_0 and u be as in Theorem 1.1. Suppose that the additional conditions of u are satisfied*

$$\partial_3 u_i \in L^2(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)), \quad i = 1, 2, \quad (1.17)$$

and

$$\partial_3 u_3 \in L^{\frac{4}{-5s+2}}(0, T; \dot{B}_{\infty, \infty}^{-s}(\mathbb{R}^3)) \text{ with } 0 < s < \frac{2}{5}. \quad (1.18)$$

Then u is regular.

Theorem 1.7. *Let u_0 and u be as in Theorem 1.1. Suppose that u the additional condition*

$$\partial_3 u_3 \in L^{\frac{24}{-29s+8}}(0, T; \dot{B}_{\infty, \infty}^{-s}(\mathbb{R}^3)) \text{ with } 0 < s < \frac{8}{29}. \quad (1.19)$$

Then u is regular.

Remark 1.8. By the embedding (1.16), we know that the condition (1.17) is corresponding to the endpoint case of the Prodi-Serrin conditions in the class of $L^q(0, T; L^p(\mathbb{R}^3))$ with $p = 3$ and $q = 2$, which is consistent with (1.2). While the condition (1.18) is in consistent with (1.2), however, we give a same range of q with $2 < q < \infty$ when $0 < s < \frac{2}{5}$. Furthermore, if we provide the sufficient condition, only in terms of $\partial_3 u_3$, we shall have a more strict condition, which is shown in Theorem 1.7. One can see the corresponding q varies from 3 to ∞ .

For the convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space \mathbb{R}^3 (see, for example, [7], [14]). There exists a positive constant C such that

$$\begin{aligned} \|u\|_r &\leq C \|u\|_2^{\frac{6-r}{2r}} \|\partial_1 u\|_2^{\frac{r-2}{2r}} \|\partial_2 u\|_2^{\frac{r-2}{2r}} \|\partial_3 u\|_2^{\frac{r-2}{2r}} \\ &\leq C \|u\|_2^{\frac{6-r}{2r}} \|\nabla u\|_2^{\frac{3(r-2)}{2r}}, \end{aligned} \quad (1.20)$$

for every $u \in H^1(\mathbb{R}^3)$ and every $r \in [2, 6]$, where C is a constant depending only on r . Taking $\nabla \operatorname{div}$ on both sides of (1.1) for smooth $(u; p)$, one can obtain

$$-\Delta(\nabla p) = \sum_{i,j}^3 \partial_i \partial_j (\nabla(u_i u_j)),$$

therefore, the Calderon-Zygmund inequality in \mathbb{R}^3 (see [23])

$$\|\nabla p\|_q \leq C \|\nabla u\|_q \|u\|_q, \quad 1 < q < \infty, \quad (1.21)$$

holds, where C is a positive constant depending only on q . And there is another estimates for the pressure

$$\|p\|_q \leq C \|u\|_{2q}^2, \quad 1 < q < \infty, \quad (1.22)$$

Recall also that if $\operatorname{div} u = 0$ then the vorticity $\omega = \operatorname{curl} u = \nabla \times u$ has the following estimates (see [18]):

$$C \|\omega\|_q \leq \|\nabla u\|_q \leq C(q) \|\omega\|_q, \quad 1 < q < \infty. \quad (1.23)$$

Moreover, if $\operatorname{div} u = 0$, the expression

$$\Delta u = \nabla(\operatorname{div} u) - \nabla \times (\nabla \times u) \quad (1.24)$$

can be reduced to

$$\Delta u = -\nabla \times (\nabla \times u) = -\nabla \times \omega. \quad (1.25)$$

On the other hand, note that $\operatorname{div} \omega = 0$, applying (1.23), we have

$$C \|\nabla \times \omega\|_q \leq \|\nabla \omega\|_q \leq C(q) \|\nabla \times \omega\|_q, \quad 1 < q < \infty. \quad (1.26)$$

Therefore, by (1.23) and (1.25), we have

$$C \|\Delta u\|_q \leq \|\nabla \omega\|_q \leq C(q) \|\Delta u\|_q, \quad 1 < q < \infty. \quad (1.27)$$

2. PRELIMINARIES

We begin this section with some notations and Lemmas, which is useful for us to prove the main results. In order to define Besov spaces, we first introduce the Littlewood-Paley decomposition theory. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function, given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transformation $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx,$$

and its inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ is defined by

$$\check{f}(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} f(\xi) d\xi.$$

More generally, the Fourier transform of any $f \in \mathcal{S}'(\mathbb{R}^3)$, the space of tempered distributions, is given by

$$\langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle,$$

for any $g \in \mathcal{S}(\mathbb{R}^3)$. The Fourier transform is a bounded linear bijection from \mathcal{S}' to \mathcal{S}' whose inverse is also bounded. We fix the notation

$$\mathcal{S}_h = \{\phi \in \mathcal{S}, \int_{\mathbb{R}^3} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \dots\}.$$

Its dual is given by

$$\mathcal{S}'_h = \mathcal{S}' / \mathcal{S}_h^\perp = \mathcal{S}' / \mathcal{P},$$

where \mathcal{P} is the space of polynomial. In other words, two distributions in \mathcal{S}'_h are identified as the same if their difference is a polynomial. Let us choose two nonnegative radial functions

$\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported in $\mathfrak{B} = \{\xi \in \mathbb{R}^3 : |\xi| \leq 4/3\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^3 : 3/4 \leq |\xi| \leq 8/3\}$ respectively, such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^3 \setminus \{0\},$$

and

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^3.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, and then we define the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator \dot{S}_j as follows:

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x-y) dy,$$

and

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) u(x-y) dy.$$

Informally, $\dot{\Delta}_j$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while \dot{S}_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. And one can easily verify that $\dot{\Delta}_j \dot{\Delta}_k f = 0$ if $|j-k| \geq 2$. Especially for any $f \in L^2(\mathbb{R}^3)$, we have the Littlewood-Paley decomposition:

$$f = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j f.$$

We now give the definitions of Besov spaces. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^3)$ is defined by the full-dyadic decomposition. We say that $f \in \dot{B}_{p,q}^s(\mathbb{R}^3)$, if $f \in S'_h$ and

$$\sum_{j=-\infty}^{+\infty} (2^{js} \|\dot{\Delta}_j f\|_{L^p})^q < \infty,$$

with the norm

$$\|f\|_{\dot{B}_{p,q}^s} = \begin{cases} \left(\sum_{j=-\infty}^{+\infty} 2^{qjs} \|\dot{\Delta}_j f\|_{L^p}^q \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}, & q = \infty. \end{cases}$$

It is of interest to note that the homogeneous Besov space $\dot{B}_{2,2}^s(\mathbb{R}^3)$ is equivalent to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$. The following Bernstein inequalities will be used in the next section.

Lemma 2.1. (see [1]) *Let \mathfrak{B} be a ball and \mathfrak{C} an annulus. A constant C exists such that for any nonnegative integer k , and couple (p, q) in $[1, \infty]^2$ with $1 \leq p \leq q$, and any function u of $L^p(\mathbb{R}^d)$, we have*

$$\text{Supp } \hat{u} \subset \lambda \mathfrak{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \quad (2.1)$$

$$\text{Supp } \hat{u} \subset \lambda \mathfrak{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \quad (2.2)$$

Lemma 2.2. (see [1]) *Let $1 \leq q < p < \infty$ and α be a positive real number. A constant C exists such that*

$$\|f\|_{L^p} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{1-\theta} \|f\|_{\dot{B}_{q,q}^\beta}^\theta, \text{ with } \beta = \alpha \left(\frac{p}{q} - 1 \right) \text{ and } \theta = \frac{q}{p}. \quad (2.3)$$

In particular, for $\beta = 1, q = 2$ and $p = 4$, we get $\alpha = 1$ and

$$\|f\|_{L^4} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-1}}^{1/2} \|f\|_{\dot{H}^1}^{1/2}, \quad (2.4)$$

and further, if we give the suitable values to parameters β, q, p, α , we get other inequalities, for example

$$\|f\|_{L^6} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-\frac{1}{2}}}^{2/3} \|f\|_{\dot{H}^1}^{1/3}, \quad \|f\|_{L^3} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-2}}^{1/3} \|f\|_{\dot{H}^1}^{2/3}. \quad (2.5)$$

Lemma 2.3. (see [1]) *A constant C exists which satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0, 1)$, for any $(p, r) \in [1, \infty]^2$ and any $f \in \dot{B}_{p,r}^{s_1} \cap \dot{B}_{p,r}^{s_2}$, then we have*

$$\|f\|_{\dot{B}_{p,r}^{\theta s_1 + (1-\theta)s_2}} \leq C \|f\|_{\dot{B}_{p,r}^{s_1}}^\theta \|f\|_{\dot{B}_{p,r}^{s_2}}^{(1-\theta)}. \quad (2.6)$$

3. PROOF OF MAIN RESULTS

In this section, under the assumptions of the Theorem 1.1, Theorem 1.3 or Theorem 1.6 in Section 1 respectively, we prove our main results. First of all, we note that, by the energy inequality, for Leray-Hopf weak solutions, we have (see, for example, [22], [24] for detail)

$$\|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq K_1, \quad (3.1)$$

for all $0 < t < T$, where $K_1 = \|u_0\|_{L^2}^2$.

It is well known that there exists a unique strong solution u local in time if $u_0 \in V$. In addition, this strong solution $u \in C([0, T^*); V) \cap L^2(0, T^*; H^2(\mathbb{R}^3))$ is the only weak solution with the initial datum u_0 , where $(0, T^*)$ is the maximal interval of existence of the unique strong solution. If $T^* \geq T$, then there is nothing to prove. If, on the other hand, $T^* < T$, then our strategy is to show that the H^1 norm of this strong solution is bounded uniformly in time over the interval $(0, T^*)$, provided additional conditions in Theorem 1.1, Theorem 1.3 or Theorem 1.6 in Section 1 are valid. As a result the interval $(0, T^*)$ can not be a maximal interval of existence, and consequently $T^* \geq T$, which concludes our proof.

In order to prove the H^1 norm of the strong solution u is bounded on interval $(0, T^*)$, combining with the energy equality (3.1), it is sufficient to prove

$$\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C, \quad \forall t \in (0, T^*) \quad (3.2)$$

where the constant C depends on T, K_1 .

Proof of Theorem 1.1 Taking the curl on (1.1), we obtain

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = 0.$$

We taking the inner product of above inequality with ω in $L^2(\mathbb{R}^3)$, and by using of the Hölder's and Young's inequalities, as well as (1.23), (1.27) and (2.4), we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 &= \int_{\mathbb{R}^3} (\omega \cdot \nabla) u \cdot \omega dx \\
&= \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \omega_i \partial_i u_j \omega_j dx \\
&\leq C \|\omega\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\leq C \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\leq C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} \\
&\leq C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\Delta u\|_{L^2}^2 \\
&\leq C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \omega\|_{L^2}^2.
\end{aligned} \tag{3.3}$$

Absorbing the first term in right hand side and integrating the above inequality, we obtain

$$\|\omega\|_{L^2}^2 + \nu \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \leq \|\omega_0\|_{L^2}^2 + C \int_0^t \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 d\tau. \tag{3.4}$$

Therefore, by Gronwall's inequality, one has

$$\|\omega\|_{L^2}^2 + \nu \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \leq \|\omega_0\|_{L^2}^2 \exp \left(C \int_0^t \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 d\tau \right).$$

By using of Gronwall's inequality and condition (1.10), we have

$$\omega \in L^\infty(0, T^*; L^2(\mathbb{R}^3)) \cap L^2(0, T^*; H^1(\mathbb{R}^3)).$$

Therefore, by (1.23) and (1.27), we get the H^1 norm of the strong solution u is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.3 Firstly, we deal with (i). Taking the inner product of the equation (1.1) with $-\Delta_h u$ in $L^2(\mathbb{R}^3)$, By integrating by parts a few times and using the incompressibility condition, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 &= \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \cdot \Delta_h u dx \\
&= - \int_{\mathbb{R}^3} \sum_{k,j=1}^3 \sum_{i=1}^2 \partial_j u_k \partial_i u_j \partial_i u_k dx \\
&\leq C \int_{\mathbb{R}^3} |\nabla u| |\nabla_h u|^2 dx.
\end{aligned} \tag{3.5}$$

Applying Hölder's and Young's inequalities, as well as (2.4), we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 &\leq \|\nabla_h u\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\leq \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2} \\
&\leq \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla_h \nabla u\|_{L^2}^2.
\end{aligned} \tag{3.6}$$

Absorbing the first term in right hand side and integrating the above inequality, we obtain

$$\|\nabla_h u\|_{L^2}^2 + \nu \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \leq \|\nabla_h u(0)\|_{L^2}^2 + \int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 d\tau. \tag{3.7}$$

Next, we also use $-\Delta u$ as test function, and get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \\
&= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx \\
&= \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \Delta_h u_j dx + \sum_{i=1}^2 \sum_{j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta u_j dx + \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_{33} u_j dx \\
&= L_1(t) + L_2(t) + L_3(t)
\end{aligned}$$

The calculation has been shown in [26], for the convenience of readers, we list it below. By integrating by parts a few times and using the incompressibility condition, we get $L_1(t), L_2(t), L_3(t)$ as follows

$$\begin{aligned}
L_1(t) &= - \sum_{j=1}^3 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx - \sum_{j=1}^3 \sum_{k=1}^2 \int_{\mathbb{R}^3} u_3 \partial_{3k} u_j \partial_k u_j dx \\
&= - \sum_{j=1}^3 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx + \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_k u_j \partial_k u_j dx, \\
L_2(t) &= - \sum_{i=1}^2 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx - \sum_{i=1}^2 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} u_i \partial_{ik} u_j \partial_k u_j dx \\
&= - \sum_{i=1}^2 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_i u_i \partial_k u_j \partial_k u_j dx, \\
L_3(t) &= - \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx = \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^3} (\partial_1 u_1 + \partial_2 u_2) \partial_3 u_j \partial_3 u_j dx.
\end{aligned}$$

Therefore, by (1.20) and Hölder's inequalities, for every i ($i = 1, 2, 3$) we have

$$\begin{aligned}
|L_i(t)| &\leq C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx \\
&\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2 \\
&\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}},
\end{aligned} \tag{3.8}$$

and hence we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}}. \tag{3.9}$$

Integrating (3.9), applying Hölder's inequality and combining (3.7) and (3.8), we obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\leq \|\nabla u(0)\|_{L^2}^2 + (\sup_{0 \leq s \leq t} \|\nabla_h u\|_{L^2}) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
&\quad \times \left(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
&\leq \|\nabla u(0)\|_{L^2}^2 + C \left(\int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 d\tau \right) \times \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
&\quad + \|\nabla_h u(0)\|_{L^2}^2 \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}.
\end{aligned} \tag{3.10}$$

By using of the Hölder's and Young's inequalities, it follows that

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C\|\nabla_h u(0)\|_{L^2}^{8/3} \\
& \quad + C \left(\int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 d\tau \right)^{4/3} \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C\|\nabla_h u(0)\|_{L^2}^{8/3} \\
& \quad + C \left(\int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^{8/3} \|\nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{1/4}.
\end{aligned} \tag{3.11}$$

Thanks again to the energy inequality, we get

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau & \leq \|\nabla u(0)\|_{L^2}^2 + C\|\nabla_h u(0)\|_{L^2}^{8/3} \\
& \quad + C \int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^{8/3} \|\nabla u\|_{L^2}^2 d\tau.
\end{aligned} \tag{3.12}$$

Therefore, by using of Gronwall's inequality, we finally obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \left(\|\nabla u(0)\|_{L^2}^2 + C\|\nabla_h u(0)\|_{L^2}^{8/3} \right) \exp \left(C \int_0^t \|\nabla_h u\|_{\dot{B}_{\infty,\infty}^{-1}}^{8/3} d\tau \right),
\end{aligned}$$

by condition (1.12), we get the H^1 norm of the strong solution u is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of (i).

Now we prove (ii). Taking the inner product of the equation (1.1) with $-\Delta_h u$ in $L^2(\mathbb{R}^3)$, we have (see [4] for detail)

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 & = \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \cdot \Delta_h u dx \\
& \leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| dx
\end{aligned} \tag{3.13}$$

By using of the Littlewood-Paley decomposition, we decompose u_3 as follows:

$$u_3 = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j u_3 = \sum_{j < [\sigma]} \dot{\Delta}_j u_3 + \sum_{j \geq [\sigma]+1} \dot{\Delta}_j u_3, \tag{3.14}$$

where σ is a real number determined later, and $[\cdot]$ denotes the integer part of σ . Therefore, (3.13) becomes

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq I_1(t) + I_2(t), \tag{3.15}$$

with

$$I_1(t) = C \sum_{j < [\sigma]} \int_{\mathbb{R}^3} |\dot{\Delta}_j u_3| |\nabla u| |\nabla_h \nabla u| dx,$$

$$I_2(t) = C \sum_{j \geq [\sigma]+1} \int_{\mathbb{R}^3} |\dot{\Delta}_j u_3| |\nabla u| |\nabla_h \nabla u| dx.$$

In what following, we estimate $I_1(t)$ and $I_2(t)$. For $I_1(t)$, by using of the Hölder's and Young's inequalities, as well as Lemma 2.1, we have

$$\begin{aligned}
I_1(t) &\leq C \sum_{j < [\sigma]} \|\dot{\Delta}_j u_3\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \\
&\leq C \left(\sum_{j < [\sigma]} 2^{\frac{3}{2}j} \right) \|u_3\|_{L^2} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \\
&\leq C 2^{\frac{3}{2}[\sigma]} \|u_3\|_{L^2} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2}. \\
&\leq C 2^{3\sigma} \|u_3\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{4} \|\nabla_h \nabla u\|_{L^2}^2,
\end{aligned} \tag{3.16}$$

the last inequality, we use the fact that $[\sigma] \leq \sigma$. As to $I_2(t)$, we take the same strategy to $I_1(t)$, by the definition of norm of the Besov space, for any $0 < \varepsilon < 1$, we have

$$\begin{aligned}
I_2(t) &\leq C \sum_{j \geq [\sigma]+1} \|\dot{\Delta}_j u_3\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \\
&\leq C \sum_{j \geq [\sigma]+1} 2^{-\varepsilon j} 2^{(-1+\varepsilon)j} \|\dot{\Delta}_j \nabla u_3\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \\
&\leq C \left(\sum_{j \geq [\sigma]+1} 2^{-\varepsilon j} \right) \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}} \|\nabla u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \\
&\leq C 2^{-2\varepsilon([\sigma]+1)} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{4} \|\nabla_h \nabla u\|_{L^2}^2. \\
&\leq C 2^{-2\varepsilon\sigma} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{4} \|\nabla_h \nabla u\|_{L^2}^2,
\end{aligned} \tag{3.17}$$

the last inequality, we use the fact that $\sigma < [\sigma] + 1$. Inserting (3.16) and (3.17) into (3.13) to obtain

$$\frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq C 2^{-2\varepsilon\sigma} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^2 \|\nabla u\|_{L^2}^2 + C 2^{3\sigma} \|u_3\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \tag{3.18}$$

Now, we choose σ such that

$$2^{\frac{3}{2}\sigma} \|u_3\|_{L^2} \|\nabla u\|_{L^2} = 2^{-\varepsilon\sigma} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}} \|\nabla u\|_{L^2},$$

then we have

$$C 2^{-\varepsilon\sigma} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}} \|\nabla u\|_{L^2} \leq C \|u_3\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}} \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}.$$

Integrating (3.18), combing above two inequalities and the energy inequality (3.1) we have

$$\|\nabla_h u\|_{L^2}^2 + \nu \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \leq \|\nabla_h u(0)\|_{L^2}^2 + C + C \int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{3}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}^2 d\tau. \tag{3.19}$$

Integrating (3.9), applying Hölder's inequality and combining (3.1) and (3.19), we obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \|\nabla u(0)\|_{L^2}^2 + (\sup_{0 \leq s \leq t} \|\nabla_h u\|_{L^2}) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \quad \times \left(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C \left(\int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{3}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}^2 d\tau \right) \times \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \quad + (\|\nabla_h u(0)\|_{L^2}^2 + C) \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}.
\end{aligned} \tag{3.20}$$

By using of the Hölder's and Young's inequalities, it follows that

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C \left(\|\nabla_h u(0)\|_{L^2}^{8/3} + 1 \right) \\
& \quad + C \left(\int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{3}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}^2 d\tau \right)^{4/3} \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C \left(\|\nabla_h u(0)\|_{L^2}^{8/3} + 1 \right) \\
& \quad + C \left(\int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{4}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{1/4}.
\end{aligned} \tag{3.21}$$

Thanks again to the energy inequality, we get

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau & \leq \|\nabla u(0)\|_{L^2}^2 + C \left(\|\nabla_h u(0)\|_{L^2}^{8/3} + 1 \right) \\
& \quad + C \int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-1+\varepsilon}}^{\frac{4}{\frac{3}{2}+\varepsilon}} \|\nabla u\|_{L^2}^2 d\tau.
\end{aligned} \tag{3.22}$$

If we set $s = 1 - \varepsilon$, then we have

$$\frac{4}{\frac{3}{2} + \varepsilon} = \frac{8}{5 - 2s} \text{ with } 0 < s < 1.$$

Therefore, by using of Gronwall's inequality, we finally obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \left(\|\nabla u(0)\|_{L^2}^2 + C(\|\nabla_h u(0)\|_{L^2}^{8/3} + 1) \right) \exp \left(C \int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{8}{5-2s}} d\tau \right),
\end{aligned}$$

by condition (1.13), we get the H^1 norm of the strong solution u is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of Theorem 1.3.

Proof of Theorem 1.6 We also split the proof into two steps. Recalling that in Theorem 1.2, the vorticity ω satisfies

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 = \int_{\mathbb{R}^3} (\omega \cdot \nabla) u \cdot \omega dx = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \omega_i \partial_i u_j \omega_j dx.$$

Since $\omega = (\omega_1, \omega_2, \omega_3) = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$, we put the detail computation to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} \partial_3 u_2 \partial_3 u_2 \partial_1 u_1 dx - \int_{\mathbb{R}^3} \partial_1 u_2 \partial_3 u_2 \partial_3 u_1 dx \\
&+ \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_1 \partial_2 u_2 dx - \int_{\mathbb{R}^3} \partial_2 u_1 \partial_3 u_2 \partial_3 u_1 dx \\
&- \int_{\mathbb{R}^3} u_3 \{ \partial_2 (\partial_2 u_3 \partial_1 u_1 - 2 \partial_3 u_2 \partial_1 u_1 + \partial_1 u_2 \partial_3 u_1 \\
&\quad - \partial_1 u_2 \partial_1 u_3 - \partial_2 u_1 \partial_1 u_3 + \partial_1 u_1 \partial_3 u_1) \} dx \\
&- \int_{\mathbb{R}^3} u_3 \{ \partial_1 (\partial_3 u_2 \partial_1 u_2 + \partial_1 u_3 \partial_2 u_1 - 2 \partial_2 u_2 \partial_3 u_1 \\
&\quad + \partial_2 u_2 \partial_1 u_3 - \partial_3 u_2 \partial_1 u_3 + \partial_2 u_1 \partial_3 u_2) \} dx \\
&- \int_{\mathbb{R}^3} \sum_{i=1}^3 \omega_i u_3 \partial_i \omega_3 dx,
\end{aligned} \tag{3.23}$$

then, by Young's inequality, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 \\
&\leq \int_{\mathbb{R}^3} |\partial_3 u_2|^2 |\partial_1 u_1| dx + \int_{\mathbb{R}^3} |\partial_3 u_2| |\partial_1 u_2| |\partial_3 u_1| dx \\
&+ \int_{\mathbb{R}^3} |\partial_3 u_1|^2 |\partial_2 u_2| dx + \int_{\mathbb{R}^3} |\partial_3 u_2| |\partial_2 u_1| |\partial_3 u_1| dx \\
&+ C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\Delta u| dx. \\
&\leq C \int_{\mathbb{R}^3} |\partial_3 u_2|^2 |\nabla u| dx + C \int_{\mathbb{R}^3} |\partial_3 u_1|^2 |\nabla u| dx \\
&+ C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\Delta u| dx. \\
&= K_1(t) + K_2(t) + K_3(t).
\end{aligned} \tag{3.24}$$

Next, we estimate $K_i(t)$ one by one, $i = 1, 2, 3$. Applying Hölder's and Young's inequalities, as well as (1.23), (1.27) and (2.4), we have

$$\begin{aligned}
K_1(t) &= C \int_{\mathbb{R}^3} |\partial_3 u_2|^2 |\nabla u| dx \\
&\leq C \|\partial_3 u_2\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\leq C \|\partial_3 u_2\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla \partial_3 u_2\|_{L^2} \|\nabla u\|_{L^2} \\
&\leq C \|\partial_3 u_2\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{8} \|\Delta u\|_{L^2}^2 \\
&\leq C \|\partial_3 u_2\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 + \frac{\nu}{8} \|\nabla \omega\|_{L^2}^2,
\end{aligned} \tag{3.25}$$

$$K_2(t) = C \int_{\mathbb{R}^3} |\partial_3 u_1|^2 |\nabla u| dx \leq C \|\partial_3 u_1\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 + \frac{\nu}{8} \|\nabla \omega\|_{L^2}^2. \tag{3.26}$$

Now, we estimate $K_3(t)$, again, applying Hölder's and Young's inequalities, as well as (1.20) and (1.27), we obtain

$$\begin{aligned}
K_3(t) &= C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\Delta u| dx \\
&\leq C \|u_3\|_{L^q} \|\nabla u\|_{L^p} \|\Delta u\|_{L^2} \\
&\leq C \|u_3\|_{L^q} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\Delta u\|_{L^2}^{\frac{5p-6}{2p}} \\
&\leq C \|u_3\|_{L^q}^{\frac{4p}{6-p}} \|\nabla u\|_{L^2}^2 + \frac{\nu}{16} \|\Delta u\|_{L^2}^2 \\
&= C \|u_3\|_{L^q}^{\frac{2q}{q-3}} \|\nabla u\|_{L^2}^2 + \frac{\nu}{16} \|\nabla \omega\|_{L^2}^2,
\end{aligned} \tag{3.27}$$

where p and q satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \text{ with } 2 \leq p < 6, q > 3.$$

For $\|u_3\|_{L^q}$, we have the following estimate. We use $|u_3|^{q-2}u_3$ as test function in the equation (1.1) for u_3 . By using of Gagliardo-Nirenberg and Hölder's inequalities, and applying the inequality (1.22), we have

$$\begin{aligned}
\frac{1}{q} \frac{d}{dt} \|u_3\|_{L^q}^q + C\nu \|\nabla |u_3|^{\frac{q}{2}}\|_{L^2}^2 &= - \int_{\mathbb{R}^3} \partial_3 p |u_3|^{q-2} u_3 dx \\
&\leq C \int_{\mathbb{R}^3} |p| |u_3|^{q-2} |\partial_3 u_3| dx \\
&\leq C \|p\|_{L^\mu} \|u_3\|_{L^q}^{q-2} \|\partial_3 u_3\|_{L^\beta} \\
&\leq C \|u\|_{L^{2\mu}}^2 \|u_3\|_{L^q}^{q-2} \|\partial_3 u_3\|_{L^\beta} \\
&\leq C \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|u_3\|_{L^q}^{q-2} \|\partial_3 u_3\|_{L^\beta}.
\end{aligned} \tag{3.28}$$

The parameters in above inequality satisfy

$$\frac{1}{\mu} + \frac{1}{\beta} + \frac{q-2}{q} = 1, \text{ with } 1 \leq \mu \leq 3, \beta > 7. \tag{3.29}$$

By (1.27), (2.3) and the energy inequality, The inequality (3.28) implies that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u_3\|_{L^q}^2 &\leq C \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|\partial_3 u_3\|_{L^\beta} \\
&\leq C \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{1-\frac{2}{\beta}}}^{1-\frac{2}{\beta}} \|\nabla \partial_3 u_3\|_{L^2}^{\frac{2}{\beta}} \\
&\leq C \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{1-\frac{2}{\beta}}}^{1-\frac{2}{\beta}} \|\Delta u\|_{L^2}^{\frac{2}{\beta}},
\end{aligned} \tag{3.30}$$

Integrating (3.30), applying (3.1) and Hölder's inequality, we have

$$\begin{aligned}
\|u_3\|_{L^q}^2 &\leq \|u_3(0)\|_{L^q}^2 + C \int_0^t \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{1-\frac{2}{\beta}}}^{1-\frac{2}{\beta}} \|\Delta u\|_{L^2}^{\frac{2}{\beta}} d\tau \\
&\leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{1-\frac{2}{\beta}}}^{(1-\frac{2}{\beta})(\frac{\beta}{\beta-1})} \|\nabla u\|_{L^2}^{\frac{3\beta(\mu-1)}{\mu(\beta-1)}} d\tau \right)^{\frac{\beta-1}{\beta}} \\
&\quad \times \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{\beta}} + \|u_3(0)\|_{L^q}^2
\end{aligned} \tag{3.31}$$

We choose

$$\mu = \frac{3\beta}{\beta+2}, \quad (3.32)$$

then

$$\frac{3\beta(\mu-1)}{\mu(\beta-1)} = 2.$$

Combing (3.27) and (3.31), by energy inequality, and using Hölder's and Young's inequalities, we get

$$\begin{aligned} \int_0^t K_3(\tau) d\tau &\leq C \left(\sup_{0 \leq \tau \leq t} \|u_3\|_{L^q}^2 \right)^{\frac{q}{q-3}} \int_0^t \|\nabla u\|_{L^2}^2 d\tau + \frac{\nu}{16} \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \\ &\leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{(1-\frac{2}{\beta})(\frac{\beta}{\beta-1})} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{q(\beta-1)}{\beta(q-3)}} \left(\int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \right)^{\frac{q}{\beta(q-3)}} \\ &\quad + C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} + \frac{\nu}{16} \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \\ &\leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{(1-\frac{2}{\beta})(\frac{\beta}{\beta-1})} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{q(\beta-1)}{\beta(q-3)-q}} + C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} \\ &\quad + \frac{\nu}{8} \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \\ &\leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{\frac{q(\beta-2)}{\beta(q-3)-q}} \|\nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{3\beta}{q(\beta-1)}} \\ &\quad + C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} + \frac{\nu}{8} \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \\ &\leq C \int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{\frac{q(\beta-1)}{\beta(q-3)-q}} \|\omega\|_{L^2}^2 d\tau + C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} \\ &\quad + \frac{\nu}{8} \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau, \end{aligned} \quad (3.33)$$

in above inequality, we note that q and β satisfy the condition

$$\frac{q}{\beta(q-3)} < 1.$$

Integrating (3.24), combining (3.25), (3.26), (3.41), and applying Young's inequality, we have

$$\begin{aligned} \|\omega\|_{L^2}^2 + \nu \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau &\leq C \int_0^t \|\partial_3 u_2\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 d\tau + C \int_0^t \|\partial_3 u_1\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\omega\|_{L^2}^2 d\tau \\ &\quad + C \int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{\frac{q(\beta-1)}{\beta(q-3)-q}} \|\omega\|_{L^2}^2 d\tau + C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} + \|\omega(0)\|_{L^2}^2. \end{aligned} \quad (3.34)$$

If we denote $s = \frac{2}{\beta-2}$, then by (3.29) and (3.32), we have

$$\frac{q(\beta-1)}{\beta(q-3)-q} = \frac{4}{-5s+2} \text{ with } 0 < s < \frac{2}{5}$$

By using of Gronwall's inequality, we obtain

$$\begin{aligned}
& \|\omega\|_{L^2}^2 + \nu \int_0^t \|\nabla \omega\|_{L^2}^2 d\tau \\
& \leq \left(C \|u_3(0)\|_{L^q}^{\frac{2q}{q-3}} + \|\omega(0)\|_{L^2}^2 \right) \exp \left(C \int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{-\frac{4}{5s+2}}}^2 d\tau \right), \\
& \quad \times \exp \left(C \int_0^t \|\partial_3 u_1\|_{\dot{B}_{\infty,\infty}^{-1}}^2 d\tau \right) \exp \left(C \int_0^t \|\partial_3 u_2\|_{\dot{B}_{\infty,\infty}^{-1}}^2 d\tau \right)
\end{aligned} \tag{3.35}$$

by the condition (1.17) and (1.18), we have

$$\omega \in L^\infty(0, T^*; L^2(\mathbb{R}^3)) \cap L^2(0, T^*; H^1(\mathbb{R}^3)).$$

Therefore, the H^1 norm of the strong solution u is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of Theorem 1.6.

Proof of Theorem 1.7 Firstly, we begin with (3.13), and by (1.20), we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \\
& \leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| dx \\
& \leq C \|u_3\|_{L^q} \|\nabla u\|_{L^p} \|\nabla_h \nabla u\|_{L^2} \\
& \leq C \|u_3\|_{L^q} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\Delta u\|_{L^2}^{\frac{p-2}{2p}} \|\nabla_h \nabla u\|_{L^2}^{\frac{2(p-1)}{p}} \\
& \leq C \|u_3\|_{L^q}^p \|\nabla u\|_{L^2}^{\frac{6-p}{2}} \|\Delta u\|_{L^2}^{\frac{p-2}{2}} + \frac{\nu}{2} \|\nabla_h \nabla u\|_{L^2}^2,
\end{aligned} \tag{3.36}$$

where p and q satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \text{ with } 2 \leq p < 5, q > \frac{10}{3}.$$

Therefore, integrating above inequality and using Hölder's inequality, it follows that

$$\begin{aligned}
& \|\nabla_h u\|_{L^2}^2 + \nu \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \\
& \leq \|\nabla_h u(0)\|_{L^2}^2 + \int_0^t \|u_3\|_{L^q}^p \|\nabla u\|_{L^2}^{\frac{6-p}{2}} \|\Delta u\|_{L^2}^{\frac{p-2}{2}} d\tau \\
& \leq \|\nabla_h u(0)\|_{L^2}^2 + \left(\int_0^t \|u_3\|_{L^q}^{\frac{4p}{6-p}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{6-p}{4}} \left(\int_0^t \|\Delta u\|_{L^2}^{\frac{p-2}{2}} d\tau \right)^{\frac{p-2}{4}}
\end{aligned} \tag{3.37}$$

For $\|u_3\|_{L^q}$, we have the same estimate to (3.31), in which the parameters satisfy (3.32) and

$$\frac{1}{\mu} + \frac{1}{\beta} + \frac{q-2}{q} = 1, \text{ with } 1 \leq \mu \leq 3, \beta > \frac{37}{4}. \tag{3.38}$$

Next, in view of (3.9), integrating (3.9), applying Hölder's and Young's inequalities and combining (3.37), we obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \|\nabla u(0)\|_{L^2}^2 + (\sup_{0 \leq s \leq t} \|\nabla_h u\|_{L^2}) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \quad \times \left(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq \|\nabla u(0)\|_{L^2}^2 + \left(\int_0^t \|u_3\|_{L^q}^{\frac{4p}{6-p}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{6-p}{4}} \left(\int_0^t \|\Delta u\|_{L^2}^{\frac{p-2}{2}} d\tau \right)^{\frac{p-1}{4}} \\
& \quad + \|\nabla_h u(0)\|_{L^2}^2 \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq \|\nabla u(0)\|_{L^2}^2 + \left(\int_0^t \|u_3\|_{L^q}^{\frac{4p}{6-p}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{6-p}{5-p}} \\
& \quad + C \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau.
\end{aligned} \tag{3.39}$$

We finally get

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \frac{3\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \|\nabla u(0)\|_{L^2}^2 + C \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} + \left(\int_0^t \|u_3\|_{L^q}^{\frac{4p}{6-p}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{6-p}{5-p}}.
\end{aligned} \tag{3.40}$$

Now, combining (3.31) and (3.40), again, by energy inequality, and using Hölder's and Young's inequalities, we get

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \frac{3\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq C (\sup_{0 \leq \tau \leq t} \|u_3\|_{L^q}^2)^{\frac{2p}{5-p}} \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{6-p}{5-p}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} \\
& \leq C (\sup_{0 \leq \tau \leq t} \|u_3\|_{L^q}^2)^{\frac{4q}{3q-10}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} \\
& \leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2-\beta}{2}}}^{(1-\frac{2}{\beta})(\frac{\beta}{\beta-1})} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{4q(\beta-1)}{\beta(3q-10)}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{4q}{\beta(3q-10)}} \\
& \quad + C \|u_3(0)\|_{L^q}^{\frac{8q}{3q-10}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} \\
& \leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2-\beta}{2}}}^{(1-\frac{2}{\beta})(\frac{\beta}{\beta-1})} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{4q(\beta-1)}{\beta(3q-10)-4q}} + C \|u_3(0)\|_{L^q}^{\frac{8q}{3q-10}} \\
& \quad + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} + \frac{\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau.
\end{aligned} \tag{3.41}$$

In above inequality, we note that β and q satisfy the additional condition

$$\frac{4q}{\beta(3q-10)} < 1.$$

By Hölder's and Young's inequalities, as well as the energy inequality, from (3.41) we have

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq C \left(\int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{\frac{4q(\beta-2)}{\beta(3q-10)-4q}} \|\nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{\beta(q+10)}{4q(\beta-1)}} \\
& \quad + C \|u_3(0)\|_{L^q}^{\frac{8q}{3q-10}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} \\
& \leq C \int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{\frac{2}{2-\beta}}}^{\frac{12(\beta-2)}{4\beta-37}} \|\nabla u\|_{L^2}^2 d\tau \\
& \quad + C \|u_3(0)\|_{L^q}^{\frac{24\beta}{9\beta-25}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}}.
\end{aligned} \tag{3.42}$$

Note that $\beta > \frac{37}{4}$, if we set $s = \frac{2}{\beta-2}$, then we have

$$\frac{12(\beta-2)}{4\beta-37} = \frac{24}{8-29s} \text{ with } 0 < s < \frac{8}{29}.$$

By using of Gronwall's inequality, we obtain

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
& \leq \left(C \|u_3(0)\|_{L^q}^{\frac{24\beta}{9\beta-25}} + \|\nabla u(0)\|_{L^2}^2 + \|\nabla_h u(0)\|_{L^2}^{\frac{8}{3}} \right) \exp \left(C \int_0^t \|\partial_3 u_3\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{24}{8-29s}} d\tau \right),
\end{aligned} \tag{3.43}$$

by the condition (1.19), the H^1 norm of the strong solution u is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of Theorem 1.7.

ACKNOWLEDGEMENT

The second author would like to thank Dr. Ting Zhang for his helpful suggestions. This work is supported by NSFC 11271322, 10931007, and Zhejiang NSF of China Z6100217.

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